IV.Cohomology of a chain complex

1. Let $\mathcal{C} = \{C_p, \partial\}$ be a chain complex of abelian groups (or R-modules) and G be an abelian group (or an R-module).

$C^p(\mathcal{C}; G) = \mathrm{Hom}(C_p, G)$: p-dimensional cochain group of $\mathcal{C}$.
(or $\mathrm{Hom}_R(C^p, G)$: p-dimensional cochain R-module of $\mathcal{C}$.)

coboundary operator $\delta : C^p \to C^{p+1}$ is the dual of $\partial : C_{p+1} \to C_p$.
$\Rightarrow \delta^2 = 0$, since $\partial^2 = 0$.

$$\to C_{p+2} \xrightarrow{\partial} C_{p+1} \xrightarrow{\partial} C_p \to \cdots, \quad \alpha : C_p \to G, \quad \alpha\circ\partial : C_{p+1} \to G$$

$\delta(\alpha) := \alpha \circ \partial$로 정의되며, 또한 $\delta^2(\alpha) := (\alpha \circ \partial) \circ \partial = \alpha \circ \partial^2 = 0$이 성립한다. 따라서,

$$\cdots \to C_{p+1} \xrightarrow{\partial} C_p \xrightarrow{\partial} C_{p-1} \xrightarrow{\partial} \cdots$$
$$\Rightarrow \cdots \leftarrow C^{p+1} \xleftarrow{\delta} C^p \xleftarrow{\delta} C^{p-1} \leftarrow \cdots : \mathcal{C}^* = \{C^p, \delta\} : \text{cochain complex.}$$

Homology of $\mathcal{C}^*$ is the cohomology of $\mathcal{C}$:
$Z^p(\mathcal{C}; G) := \ker \delta \subset C^p$, $B^p(\mathcal{C}; G) := \mathrm{im}\, \delta \subset Z^p(\mathcal{C}; G)$
$H^p(\mathcal{C}; G) := Z^p(\mathcal{C}; G)/B^p(\mathcal{C}; G)$
: cohomology of $\mathcal{C}$ with coefficient G in $\dim p$

Simplicial cohomology if $\mathcal{C}$ is the simplicial chain complex.
Singular cohomology if $\mathcal{C}$ is the singular chain complex.

2. Cohomology of augmented chain complex,

$$\cdots \to C_p \to \cdots \to C_1 \to C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is called the reduced cohomology of $\mathcal{C}$ and denoted by $\widetilde{H}^p(\mathcal{C}; G)$.

Note $\begin{cases} \widetilde{H}^p(\mathcal{C}; G) = H^p(\mathcal{C}; G) & \text{if } p > 0 \\ H^0(\mathcal{C}; G) = \widetilde{H}^0(\mathcal{C}; G) \bigoplus G & \end{cases}$ (Exercise)

3. Functorial property

A chain map $\phi : \mathcal{C} \to \mathcal{D}$ induces a chain map $\widetilde{\phi} : \mathcal{D}^* \to \mathcal{C}^*$ between the cochain complexes.

$$
\begin{array}{ccccccc}
\cdots \to C_{p+1} & \xrightarrow{\partial} & C_p \to \cdots & \Rightarrow & \cdots \leftarrow C^{p+1} & \xleftarrow[\delta]{} & C^p \leftarrow \cdots \\
\phi_{p+1}\downarrow & & \downarrow\phi_p & & \uparrow\widetilde{\phi_{p+1}} & & \widetilde{\phi_p}\uparrow \\
\cdots \to D_{p+1} & \xrightarrow{\partial} & D_p \to \cdots & & \cdots \leftarrow D^{p+1} & \xleftarrow[\delta]{} & D^p \leftarrow \cdots
\end{array}
$$

$\phi \circ \partial = \partial \circ \phi \Rightarrow \widetilde{\phi \circ \partial} = \widetilde{\partial \circ \phi} \Rightarrow \delta \circ \widetilde{\phi} = \widetilde{\phi} \circ \delta$이 성립.

Since $\widetilde{\phi}$ is a chain map, it induces a homomorphism $\phi^* : H^p(\mathcal{D}; G) \to H^p(\mathcal{C}; G)$.
Therefore,

$f : X \to Y$
$\Rightarrow f_\sharp : S_p(X) \to S_p(Y)$: singular chain map.
$\Rightarrow f^\sharp : S^p(Y) \to S^p(X)$: singular cochain map.
$\Rightarrow f^* : H^p(Y; G) \to H^p(X; G)$
,where $H^p(X; G) = H^p(S(X); G)$
: singular cohomology of X with coefficient G.

Similarly for the simplicial case.

Now

$$
\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
\uparrow\sigma & \nearrow_{f\circ\sigma = f_\sharp(\sigma)} & & & \\
\triangle & & & &
\end{array}
$$

$\Rightarrow (g \circ f)_\sharp = g_\sharp \circ f_\sharp$
$\Rightarrow (g \circ f)^\sharp = f^\sharp \circ g^\sharp$
$\Rightarrow (g \circ f)^* = f^* \circ g^*$ in H^p

And $id._\sharp = id. \Rightarrow id.^* = id.$
$\therefore H^p : \mathcal{T}op \to \mathcal{A}bel.\,groups$(or $R - \mathcal{M}od$) : contravariant functor.

4. Chain homotopy and equivalence

Let $D : \phi \simeq \psi : \mathcal{C} \to \mathcal{C}'$ be a chain homotopy, i.e., $\partial D + D\partial = \phi - \psi$

$$
\begin{array}{ccccc}
\cdots \to C_{p+1} & \xrightarrow{\partial} & C_p & \xrightarrow{\partial} & C_{p-1} \to \cdots \\
\downarrow & \swarrow_D & \phi\downarrow\psi & \swarrow_D & \\
\cdots \to C'_{p+1} & \xrightarrow[\partial]{} & C'_p & \xrightarrow[\partial]{} & C'_{p-1} \to \cdots
\end{array}
\quad \Rightarrow \quad
\begin{array}{ccccc}
\cdots \leftarrow C^{p+1} & \xleftarrow{\delta} & C^p & \leftarrow & C_{p-1} \leftarrow \cdots \\
\uparrow & \nearrow^{\widetilde{D}} & \widetilde{\psi}\uparrow\widetilde{\phi} & \nearrow_{\widetilde{D}} & \\
\cdots \leftarrow C'^{p+1} & \xleftarrow{\delta} & C'^p & \leftarrow & C'_{p-1} \leftarrow \cdots
\end{array}
$$

$\therefore \widetilde{D} \circ \widetilde{\partial} + \widetilde{\partial} \circ \widetilde{D} = \widetilde{\phi} - \widetilde{\psi}$
$\Rightarrow \delta \circ \widetilde{D} + \widetilde{D} \circ \delta = \widetilde{\phi} - \widetilde{\psi}$
$\Rightarrow \widetilde{D} : \widetilde{\phi} \simeq \widetilde{\psi}$, cochain homotopy.
In this case, $\phi^* = \psi^*$.

$\phi : \mathcal{C} \to \mathcal{C}'$, a chain homotopy equivalence
$\Rightarrow \phi_*$ and ϕ^* are isomorphisms.

따름정리 1 $f \simeq g : X \to Y \Rightarrow f_\sharp \simeq g_\sharp : S(X) \to S(Y)$.
$f^* = g^* : H^*(Y) \to H^*(X)$
Similarly for pairs, $f \simeq g : (X, A) \to (Y, B)$,
where $H^p(X, A; G) := H^p(S(X, A); G)$.

5. Long exact sequence for pairs.
Recall

$$0 \to S(A) \to S(X) \to S(X)/S(A) = S(X, A) \to 0 : \text{ s.e.s.}$$
$$\overset{\text{snake}}{\Longrightarrow} \cdots \to H_p(A) \to H_p(X) \to H_p(X, A) \xrightarrow{\partial_*} H_{p-1}(A) \to \cdots : \text{ l.e.s. of } (X, A).$$

More generally,

$$0 \to \mathcal{C} \to \mathcal{D} \to \mathcal{E} \to 0 : \text{ s.e.s.}$$
$$\overset{\text{snake}}{\Longrightarrow} \cdots \to H_p(\mathcal{C}) \to H_p(\mathcal{D}) \to H_p(\mathcal{E}) \xrightarrow{\partial_*} H_{p-1}(\mathcal{C}) \to \cdots : \text{ l.e.s.}$$

If the dual sequence of a short exact sequence is short exact, then we still obtain a long exact sequence by the snake lemma. But in general,

$$0 \to A \to B \to C \to 0 : \text{ s.e.s.}$$
$$\not\Rightarrow 0 \leftarrow A^* \leftarrow B^* \leftarrow C^* \leftarrow 0 : \text{ s.e.s.}$$

i.e., **Hom functor does not preserve short exact sequence!**

Exactness of Hom functor

정리 2 *(1)* $B \xrightarrow{g} C \to 0$ *: exact* $\Rightarrow$ *Hom*$(B, G) \xleftarrow{\tilde{g}}$ *Hom*$(C, G) \leftarrow 0$ *: exact.*
(2) $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *: exact*
$\Rightarrow$ *Hom*$(A, G) \xleftarrow{\tilde{f}}$ *Hom*$(B, G) \xleftarrow{\tilde{g}}$ *Hom*$(C, G) \leftarrow 0$ *: exact.*
(3) $0 \to A \to B \to C \to 0$ *: split exact.*
$\Rightarrow 0 \leftarrow$ *Hom*$(A, G) \xleftarrow{\tilde{f}}$ *Hom*$(B, G) \xleftarrow{\tilde{g}}$ *Hom*$(C, G) \leftarrow 0$ *: split exact.*

증명 (1) $\begin{array}{ccc} B & \xrightarrow{g} & C \\ & \searrow^{\widetilde{g}(\alpha)=\alpha\circ g} & \downarrow\alpha \\ & & G \end{array}$ Show $\widetilde{g}$ is one to one : $\widetilde{g}(\alpha) = \alpha \circ g = 0$
$\Rightarrow \alpha = 0$ since g is onto.

(2) $g \circ f = 0 \Rightarrow \widetilde{f} \circ \widetilde{g} = 0$.

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & \searrow^{0} & \downarrow\beta & \swarrow^{\bar{\beta}} & \\ & & G & & \end{array}$$

$\widetilde{f}(\beta) = \beta \circ f = 0$
$\Rightarrow \ker\beta \supset \operatorname{im} f = \ker g$ and $C \cong B/\ker g$
$\Rightarrow \beta$ induces $\bar{\beta} : C \to G$ and
$\widetilde{g}(\bar{\beta}) = \bar{\beta} \circ g = \beta$

(3)Since short exact sequence splits, there exists $p : B \to A$ such that $p \circ f = id_A$.

$$0 \longrightarrow A \underset{p}{\overset{f}{\rightleftarrows}} B \xrightarrow{g} C \longrightarrow 0$$

$\Rightarrow \widetilde{f} \circ \widetilde{p} = \widetilde{id.} = id. \Rightarrow \widetilde{f}$ is onto and Hom-sequence splits. □

Remark(1)

$$0 \to \mathbb{Z} \underset{f}{\overset{\times 2}{\to}} \mathbb{Z} \to \mathbb{Z}/2 \to 0 \qquad \text{exact}$$

$$\Rightarrow \qquad 0 \leftarrow \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \overset{\widetilde{f}}{\leftarrow} \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \leftarrow \operatorname{Hom}(\mathbb{Z}/2,\mathbb{Z}) \leftarrow 0 \qquad \text{exact(?)}$$

를 살펴보면, 우선 $\operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$이고 따라서 $\widetilde{f}$는 $\mathbb{Z}$의 1을 2로 보내는 $\times 2$인 map임을 알 수 있다. 따라서 onto가 될 수 없고, 물론 exact가 아니다.

(2) In general,

$$0 \to \mathbb{Z} \underset{f}{\overset{\times n}{\to}} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

$$\Rightarrow \qquad \operatorname{Hom}(\mathbb{Z},G) \underset{\widetilde{f}}{\overset{\times n}{\leftarrow}} \operatorname{Hom}(\mathbb{Z},G) \leftarrow \operatorname{Hom}(\mathbb{Z}/n,G) \leftarrow 0$$

$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & \mathbb{Z} \\ & \searrow^{\widetilde{f}(\alpha)} & \downarrow\alpha \\ & & G \end{array}$

a homomorphism $\alpha : \mathbb{Z} \to G$
is determined by $\alpha(1) \in G$ and
hence $\operatorname{Hom}(\mathbb{Z},G) \cong G$.
$\widetilde{f}(\alpha)(1) = \alpha(f(1)) = \alpha(n) = n\alpha(1) \Rightarrow \widetilde{f}(\alpha) = n\alpha$

$\Rightarrow \operatorname{Hom}(\mathbb{Z}/n,G) \cong \ker(G \overset{\times n}{\to} G)$
만약 $G = \mathbb{Z}/m$이면 $\operatorname{Hom}(\mathbb{Z}/n,G)$는 어떻게 되는가?(Exercise) 이들로부터 우리는 주어진 finitely generated abelian group A에 대해서 $\operatorname{Hom}(A,G)$를 계산

할 수 있다.

Return to long exact sequence:
우선 $S_p(X,A)$가 free이므로

$$0 \to S_p(A) \to S_p(X) \to S_p(X,A) \to 0$$

splits for each p, hence by the above argument,

$$0 \longleftarrow S^p(A) \longleftarrow S^p(X) \longleftarrow S^p(X,A) \longleftarrow 0$$

is exact(split) for each p. Applying snake lemma, we obtain

$$\cdots \leftarrow H^p(A;G) \leftarrow H^p(X;G) \leftarrow H^p(X,A;G) \underset{\delta^*}{\leftarrow} H^{p-1}(A;G) \leftarrow \cdots$$

Of course, it is also true for reduced cohomology.

Note. In the above l.e.s., the connecting homomorphism $\delta^*$is given as follows.

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C_{p+1} & \to & D_{p+1} & \to & E_{p+1} & \to & 0 \\ & & \partial\downarrow & & \partial\downarrow & & \partial\downarrow & & \\ 0 & \to & C_p & \longrightarrow & D_p & \longrightarrow & E_p & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array} \quad \Rightarrow \quad \begin{array}{ccccccccc} & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \leftarrow & C^{p+1} & \leftarrow & D^{p+1} & \leftarrow & E^{p+1} & \leftarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \leftarrow & C^p & \longleftarrow & D^p & \longleftarrow & E^p & \leftarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \end{array}$$

Furthermore, long exact sequence is functorial.

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{C} & \to & \mathcal{D} & \to & \mathcal{E} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{C}' & \to & \mathcal{D}' & \to & \mathcal{E}' & \to & 0 \end{array} \qquad \text{chain maps}$$

$\Rightarrow$

$$\begin{array}{ccccccccc} 0 & \leftarrow & \mathcal{C}^* & \leftarrow & \mathcal{D}^* & \leftarrow & \mathcal{E}^* & \leftarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \leftarrow & \mathcal{C}'^* & \leftarrow & \mathcal{D}'^* & \leftarrow & \mathcal{E}'^* & \leftarrow & 0 \end{array} \qquad \text{cochain maps}$$

$\Rightarrow$ Functoriality of long exact sequence follows from the earlier result. In particular, $f : (X, A) \to (Y, B) \Rightarrow f_*(f^*,\text{resp.})$ induces a homomorphism for long exact sequence of $(X, A)((Y, B),\text{resp.})$ to the long exact sequence of $(Y, B)((X, A),\text{resp.})$.

Long exact sequence of triples : $A \subset B \subset X \Rightarrow \exists$ a functorial long exact sequence,

$$\cdots \leftarrow H^p(B, A) \leftarrow H^p(X, A) \leftarrow H^p(X, B) \underset{\delta^*}{\leftarrow} H^{p-1}(B, A) \leftarrow \cdots$$

왜냐하면, 아래의 short exact sequence에서 $S(X)/S(B)$가 free이므로, sequence가 splits하고 위에서와 같이 dualize하고 snake lemma를 적용하면 되기 때문이다.

$$0 \to S(B)/S(A) \to S(X)/S(A) \to S(X)/S(B) \to 0$$

6.(Excision)

Let $\bar{U} \subset \mathring{A}$.
Then $i : (X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism $i^* : H^*(X, A) \to H^*(X - U, A - U)$.

증명 (1st proof)
Recall $i : S^{\mathcal{U}}(X) \hookrightarrow S(X)$ is a chain homotopy equivalence, where $\mathcal{U} = \{X - U, A\}$, and hence an isomorphism on cohomology.

$$\begin{array}{ccccccccc} 0 & \to & S(A) & \to & S^{\mathcal{U}}(X) & \to & S^{\mathcal{U}}(X)/S(A) & \to & 0 \\ & & \downarrow = & & \downarrow i & & \downarrow j & & \\ 0 & \to & S(A) & \to & S(X) & \to & S(X)/S(A) & \to & 0 \end{array}$$

Since $S^{\mathcal{U}}(X)/S(A)$ is free, we obtain the following diagram.

$$\begin{array}{ccccccccc}
\cdots \leftarrow & H^p(A) & \leftarrow & H^p(S^{\mathcal{U}}(X)) & \leftarrow & H^p(S^{\mathcal{U}}(X)/S(A)) & \leftarrow \cdots \\
& =\uparrow & & i^*\uparrow\cong & & j^*\uparrow & \\
\cdots \leftarrow & H^p(A) & \longleftarrow & H^p(X) & \longleftarrow & H^p(X,A) & \longleftarrow \cdots
\end{array}$$

By the 5-lemma, j^* is an isomorphism.
Furthermore, $S^{\mathcal{U}}(X)/S(A) = \frac{S(X-U)+S(A)}{S(A)} \cong \frac{S(X-U)}{S(X-U)\cap S(A)} = S(X-U, A-U)$ and this completes the proof.

□

(2nd proof)Algebraic Mapping Cone

(1)Construction

Let $f : \mathcal{C} \to \mathcal{D}$ be a chain map. Then mapping cone $Cf = \mathcal{E}$ is defined by $E_p = D_p \bigoplus C_{p-1}$ with $\partial(d,c) = (\partial d + f(c), -\partial c)$.

check $\partial^2 = 0$:

$$\partial^2 = \begin{pmatrix} \partial & f \\ 0 & -\partial \end{pmatrix} \begin{pmatrix} \partial & f \\ 0 & -\partial \end{pmatrix} = \begin{pmatrix} \partial^2 & \partial f - f\partial \\ 0 & \partial^2 \end{pmatrix} = 0$$

Now

$$0 \to D_p \xrightarrow{i} E_p \xrightarrow{p} C_{p-1} \to 0$$

where $i(d) = (d, 0)$ and $p(d,c) = c$. And

$$\begin{array}{ccccccccc}
0 \to & D_{p+1} & \to & D_{p+1} \bigoplus C_p & \longrightarrow & C_p & \longrightarrow 0 \\
& \downarrow\partial & & \downarrow\partial = \begin{smallmatrix} \partial & f \\ 0 & -\partial \end{smallmatrix} & & \downarrow -\partial & \\
0 \longrightarrow & D_p & \longrightarrow & D_p \bigoplus C_{p-1} & \to & C_{p-1} & \to 0
\end{array} \qquad \text{commutes}$$

$\Rightarrow$

$$0 \to \mathcal{D} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{C}' \to 0 \qquad \text{s.e.s. of chain complexes.}$$

where $(C'_p, \partial) = (C_{p-1}, -\partial)$. Furthermore, by applying snake lemma,

$$\Rightarrow \cdots \to H_p(\mathcal{D}) \to H_p(\mathcal{E}) \to H_p(\mathcal{C}') \to H_{p-1}(\mathcal{D}) \to \cdots$$

$$\Rightarrow \cdots \to H_p(\mathcal{D}) \to H_p(Cf) \to H_{p-1}(\mathcal{C}) \xrightarrow{f_*} H_{p-1}(\mathcal{D}) \to H_{p-1}(Cf) \to \cdots$$

$\therefore f_* : H_*(\mathcal{C}) \to H_*(\mathcal{D})$ is an isomorphism if and only if $H_*(Cf) = 0$.
Similarly for cohomology also, if $\mathcal{C}$ is free so that the above short exact sequence splits.

(2) Recall the following fact.
Let $\mathcal{C}$ be a free chain complex. Then $H_*(\mathcal{C}) = 0$(i.e. $\mathcal{C}$ is acyclic) if and only if $id. \simeq 0$(chain contractible). It easily follows from the comparison theorem.

Review of comparison theorem

$$\begin{array}{ccccccccccccc}
\cdots \to & X_n & \xrightarrow{\partial} & X_{n-1} & \xrightarrow{\partial} \cdots \to & X_1 & \xrightarrow{\partial} & X_0 & \xrightarrow{\epsilon} & A & \to 0 & & \text{augmented free chain complex over } A \\
& \downarrow & & \downarrow & & \downarrow f_1 & & \downarrow f_0 & & \downarrow \gamma & & & \\
\cdots \to & X'_n & \xrightarrow{\partial} & X'_{n-1} & \xrightarrow{\partial} \cdots \to & X'_1 & \xrightarrow{\partial} & X'_0 & \xrightarrow{\epsilon} & A' & \to 0 & & \text{resolution of } A'
\end{array}$$

$\Rightarrow$ 1.γ can be lifted to a chain map $f : X \to X'$.
2. Any two liftings are chain homotopic.

Let $f : \mathcal{C} \to \mathcal{D}$ be a chain map of free chain complexes. Then the followings are equivalent.

1. $H_*(Cf) = 0$
2. f is a chain homotopy equivalence.
3. f_* is an isomorphism.

증명 Clearly 3 implies 1 and 2 implies 3.
Remains to show 1 implies 2.

$H_*(\mathcal{E}) = 0 \Rightarrow \exists T : D_p \bigoplus C_{p-1}(= E_p) \to D_{p+1} \bigoplus C_p(= E_{p+1})$ such that $\partial T + T\partial = 1$.
Let $T_p = \begin{pmatrix} R_p & E_{p-1} \\ g_p & S_{p-1} \end{pmatrix}$, $g_p : D_p \to C_p$.

$$1 = \partial T + T\partial = \begin{pmatrix} \partial & f \\ 0 & -\partial \end{pmatrix} \begin{pmatrix} R & E \\ g & S \end{pmatrix} + \begin{pmatrix} R & E \\ g & S \end{pmatrix} \begin{pmatrix} \partial & f \\ 0 & -\partial \end{pmatrix}$$
$$= \begin{pmatrix} \partial R + fg & \partial E + fS \\ -\partial g & -\partial S \end{pmatrix} + \begin{pmatrix} R\partial & Rf + E(-\partial) \\ g\partial & gf + S(-\partial) \end{pmatrix}$$

$\Rightarrow \partial R + fg + R\partial = 1,\ -\partial g + g\partial = 0,\ -\partial S + gf - S\partial = 1$

$$\Rightarrow \left.\begin{array}{l} \partial R + R\partial = 1 - fg \\ \partial g = g\partial \end{array}\right\} \Rightarrow g \text{ is a chain map and } R \text{ is a chain homotopy} : 1 \simeq fg$$

$$\partial S + S\partial = gf - 1 \Rightarrow S : 1 \simeq gf$$

$\Rightarrow$ chain map g is a chain homotopy inverse of f and f is a chain homotopy equivalence.

□

2nd proof of excision theorem
증명 Since i_* is an isomorphism, i is a chain homotopy equivalence. Therefore, i^* is an isomorphism. □

(3) **Note** Let $\mathcal{C}$ and $\mathcal{D}$ be free chain complexes , R be a P.I.D. and $\gamma_p : H_p(\mathcal{C}) \to H_p(\mathcal{D}), \forall\, p$. Then γ is induced by a chain map ($: \mathcal{C} \to \mathcal{D}$).

따름정리 3 $H_*(\mathcal{C}) \cong H_*(\mathcal{D}) \Rightarrow H^*(\mathcal{C}) \cong H^*(\mathcal{D})$

증명 Let $\mathcal{C}$ and $\mathcal{C}'$ be free chain complexes and $\gamma_p : H_p(\mathcal{C}) \to H_p(\mathcal{C}')$ be homomorphisms, $\forall\, p$.

$$\begin{array}{ccccccccc} 0 & \to & B_p & \xrightarrow{j} & Z_p & \to & H_p & \to & 0 \quad \text{free} \\ & & \downarrow \exists\alpha & & \downarrow \exists\beta & & \downarrow \gamma_p & & \\ 0 & \to & B_p' & \to & Z_p' & \to & H_p' & \to & 0 \quad \text{acyclic} \end{array}$$

By the comparison theorem, there exist α, β such that the above diagram commutes. We want ϕ such that the following diagram commutes.

$$\begin{array}{ccccccccc} & & & & & \overset{s}{\curvearrowleft} & & & \\ 0 & \to & Z_p & \xrightarrow{i} & C_p & \underset{\partial}{\to} & B_{p-1} & \to & 0 \\ & & \downarrow \beta & & \downarrow \phi & & \downarrow \alpha & & \\ 0 & \to & Z_p' & \underset{i}{\to} & C_p' & \xrightarrow{\partial} & B_{p-1}' & \to & 0 \\ & & & & & \underset{s'}{\curvearrowleft} & & & \end{array}$$

Since B_{p-1} and B'_{p-1} are free, $C_p \cong Z_p \bigoplus D_p$ and $C'_p \cong Z'_p \bigoplus D'_p$, where $D_p = s(B_{p-1})$ and $D'_p = s'(B'_{p-1})$.
Let $\phi = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$. Then the above diagram commutes. Hence,

$$\begin{array}{ccccccc} & & & \overset{\partial}{\frown} & & & \\ C_p & \to & B_{p-1} & \xrightarrow{j} & Z_{p-1} & \xrightarrow{i} & C_{p-1} \\ \downarrow\phi & & \downarrow\alpha & & \downarrow\beta & & \downarrow\phi \\ C'_p & \to & B'_{p-1} & \to & Z'_{p-1} & \to & C'_{p-1} \\ & & & \underset{\partial}{\smile} & & & \end{array}$$

$\Rightarrow$ ϕ is a chain map and $\phi|_Z = \beta$ certainly induces γ in the first diagram. □

7. $H^p(\text{pt.}; G) = \begin{cases} G & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}$

증명 Recall

$$\cdots \to S_3(=\mathbb{Z}) \xrightarrow{0} S_2(=\mathbb{Z}) \xrightarrow{\cong} S_1(=\mathbb{Z}) \xrightarrow{0} S_0(=\mathbb{Z}) \to 0$$

$\Rightarrow$

$$\cdots \xleftarrow[\cong]{} S^3(=G) \xleftarrow[0]{} S^2(=G) \xleftarrow[\cong]{} S^1(=G) \xleftarrow[0]{} S^0(=G) \leftarrow 0$$

□

Remark

$$\left(\begin{array}{l} \text{(contravariant) functoriality property} \\ \text{long exact sequence for pairs with the existence of } \delta^* \\ \text{homotopy invariance} \\ \text{excision} \\ \text{dimension axiom 7} \end{array}\right)$$

$\Rightarrow$ Eilenberg-Steenrod axioms for (co)homology theory and unique for finite CW-pairs (Reference : Vick)

8. Let $\{X_\alpha\}$ be the family of path components of X. Then $H^p(X) \cong \prod_\alpha H^p(X_\alpha)$ for any coefficient G.

증명 $S_p(X) = \bigoplus S_p(X_\alpha), Z_p(X) = \bigoplus Z_p(X_\alpha), B_p(X) = \bigoplus B_p(X_\alpha)$ and $\mathrm{Hom}(\bigoplus A_\alpha, B) \cong \prod_\alpha \mathrm{Hom}(A_\alpha, B)$ □

9.(MV-sequence)
Same as homology case with reversed arrow of homs.
숙제 16 Check!

10. $\widetilde{H}^p(S^n; G) \cong \begin{cases} G & \text{if } p = n \\ 0 & \text{otherwise} \end{cases}$
$H^p(D^n, \partial D^n; G) \cong \begin{cases} G & \text{if } p = n \\ 0 & \text{otherwise} \end{cases}$

Same MV-sequence for adjunction space, etc.

11. Let X be a CW-complex with $\mathcal{C}(X) = \{C_p(X), \partial\}$(cellular chain complex). Then $H^p(\mathcal{C}(X); G) \cong H^p(X; G)$
증명 See 6.(3) 따름정리 1.(R:P.I.D.) □